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AFTER proposing the categorical imperative, "Act as if the maxim of thine action were to become by thy will a Universal Law of Nature," Kant tests the formula by four cases. In these he verifies his criterion indirectly, that is, by pointing out the unacceptable situations that would result from neglecting it. For example, if it were a universal law that a man in need could promise to pay back a loan even when he knows he will not be able to, then "the promise itself would become impossible, as well as the end that one might have in view in it, since no one would consider that anything was promised to him, but would ridicule all such statements as vain pretences."[1] Such an untenable consequence makes the supposition equally untenable. Therefore, Kent concludes, no such universal law is possible, but he who promises is morally bound to be sincere.

People are often more inclined to argue negatively than positively. An inept proposal is rejected more because of the difficulties it raises than because of the advantages the contrary course promises. Being a natural and easy form of argumentation, this indirect procedure has a long history in dialectics. Socrates favored it as a means of combating the self-assertiveness of the Sophists and of getting to the universal definitions

[1] I. Kant, *Fundamental Principles of the Metaphysic of Morals* (Great Books Foundation, Chicago 1949), sect. 2, pp. 43-45.

he was looking for. In fact, he uses it to such an extreme in the *Protagoras* that he seems more sophistical there than the opponent after whom Plato named the dialogue.

But the indirect proof—or disproof as it might better be called[2]—when rightly used is a cogent form of argument because of the obviousness of the principle underlying its most common form, the *reductio ad absurdum:* If a proposition implies its own denial, it is false. In Kant's example he whose promise is a self-negating pretence is simply not promising.

Sometimes the indirect method is the only kind the matter lends itself to, as for example when the terms of the judgment to be proved are related, not by a medium drawn from either by analysis, but of themselves, thus forming what Aristotle calls an immediate truth.[3] The procedure then follows the following pattern: to prove judgment *p* assume that its contradictory — *p* or non-*p* is true and draw out the implications of the assumption. If they are absurd, — *p* is excluded. No hypothesis whose implications are impossible can be true. But if — *p* is false, then *p* must be true, since of two contradictories one is true and the other is false.

In this form the argument appears on almost every page of the history of science, because each new development is tested by its implications. If these can be shown to be absurd, the hypothesis is unsound. The Phoenician inscriptions of Paraíba, Brazil are genuine, argues the archeologist, Cyrus H. Gordon, in his recent analysis, because, if they are not, the supposed nineteenth century forger would have to have known linguistic peculiarities and shadings of meaning proper to the period of Phoenician in question but discovered only in the last few years—a gratuitous and untenable implication.

The same procedure is used in metaphysics to show that being is not a genus capable of contraction. If the notion were

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[2] The common name for the argument is the "indirect proof" or the "indirect method of proof," but the term "indirect disproof" indicates its negative function: to disprove a proposition by showing that its affirmation would involve one in absurdities

[3] *Post. Anal.,* I, 3, 72b22.

contracted to its inferiors by determinative differences these latter would be being or non-being. But they cannot be being because a true difference cannot be identical with the notion it contracts; nor can they be non-being, because the logical inferiors of being must be differentiated by something. Hence the hypothesis that the notion of being is capable of contraction by true differences is false and its contradictory is true.[4]

Self-evident principles too, since their denial assumes their own truth—a principle that complements the *reductio*—allow only indirect demonstration.[5] The most obvious example is the principle of contradiction: to be and not to be in the same way at the same time is impossible. It is likewise immediately evident that if anything acts (p), it acts for some determined effect (q). If not [—(p $\supset$ q)], it acts from the supposition but at the same time does not produce this effect rather than that, that is, does not act.[6]

The method of indirect proof, therefore, depends on the principle that a false conclusion of a correct process implies falsity in the antecedent. But if the premises are held to be true, the false conclusion proves the falsity of the hypothesis that has been inserted into the sequence, which hypothesis contradicts what is to be proved.

Parmenides' apparent proof that becoming is an illusion provides a further illustration of the form the argument often takes. Let it be supposed, he argues, that something becomes. If so, it does not come out of being, because what already exists does not come to be. Yet if it really becomes, it must

[4] "That which the mind first conceives, as most known, and to which it reduces all other conceptions is being, as Avicenna says at the beginning of his Metaphysics. Hence all other concepts of the mind are formed by adding to being....some modality not explicit in the term itself." *De Ver.*, I, 1.

[5] Whitehead and Russell state the *reductio ad absurdum* as follows: *2·01 $\vdash : p \supset -p . \supset . -p$. That is, if *p* implies its own falsehood, then *p* is false. The complement of this principle is *2·18. $\vdash : -p \supset p . \supset . p$. That is, if by denying *p* one implies *p*, then *p* is true. A. Whitehead aand B. Russell, *Principia Mathematica*, 2nd ed. (Cambridge, 1963), I, 100 and 103-104.

[6] *Sum. Cont. Gent.*, III, 2.

have come out of being, because out of nothing comes nothing. Therefore nothing becomes.

In the seemingly complete disjunction reasons are given for the truth of both alternatives. Hence, if an absurdity follows—the simultaneous truth of the proposition, "Whatever becomes comes out of being," and its contradictory—it is only because a supposedly false hypothesis, "Something becomes," has been posited. The argument can be set down as follows, if *p* is taken to represent "X becomes" and *q* "X comes out of being."

(1)	p	hypothesis
(2)	$p \supset -q$	premise
(3)	$p \supset q$	premise
(4)	$-q \cdot q$	(1) (2) (3)
(5)	$-p$	(2) (3) *Per Impossibile*

Briefly, if it is assumed that something becomes (1), contradictories (4) follow, and hence the assumption (1) is alleged to be false.

Zeno of Elea, Parmenides' disciple, "composed forty proofs to demonstrate that being is one, thinking it a good thing to come to the help of his master."[7] The master's denial of phenomena seemingly so evident as becoming and multiplicity had brought him into disrepute. But both, Zeno claimed, really do involve gross absurdities if one assumes them to be verified in fact. Consequently the assumption cannot be admitted and, by opposition, Parmenides' denial must be true.

Two samples of Zeno's use of the method of indirect proof will suffice to illustrate his procedure.

I. Let the hypothesis be:

Reality is composed of many units which either have or do not have magnitude.

If each unit has magnitude it is infinitely divisible, because, no matter how much one divides, magnitude that is further

[7] F. Copleston, *A History of Philosophy* (London, 1951), I, 55, citing Proclus, *In Parmen.* 694, 23 (D 29 A 15)

divisible will remain. Consequently each unit is itself made up of an infinite number of divisible units, each with magnitude, and is therefore infinitely great. If each unit has no magnitude, nothing will have magnitude, since an aggregate of units without magnitude is itself without magnitude. Therefore reality is not infinitely great. Since under either supposition, i.e., each unit has magnitude or each unit has no magnitude, the hypothesis that reality is composed of many units leads to contradictions, it is absurd and its contradictory, Parmenides' denial of multiplicity, is true.

II. Let the hypothesis be:

There is a many, which is finite or is not finite in number.

The many must be finite in number, because there are as many things as there are, neither more nor less. Yet the many cannot be finite in number, because for units to be separate the intervention of a third thing is necessary, and so on indefinitely. Since the hypothesis, "There is a many," apparently leads to a contradiction, its contradictory is proved indirectly.

* * *

Perhaps the most celebrated use of the method of indirect proof was that by which the Pythagorean school is said to have demonstrated that the $\sqrt{2}$ is incommensurable with unity or, geometrically, that the side of a given square—which side is chosen as the unit of length—and the diagonal with the length x have no common measure. Since therefore by the Pythagorean theorem $x^2 = 1^2 + 1^2 = 2$, x or $\sqrt{2}$ cannot be a rational number.

The form of demonstration that the Pythagoreans seem actually to have used in this case is described by Aristotle as an argument *per impossibile.* It shows that, if the diagonal were commensurable with the side, the same number would be both even and not even.

For all who effect an argument *per impossibile* infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; e.g. that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this. For this we found to be reasoning *per impossibile,* viz. proving something impossible by means of an hypothesis conceded at the beginning.[8]

It was only late in the last century, when a rigorous theory of irrational numbers was worked out, that the Pythagorean doctrine of incommensurables came to be recognized for the masterpiece of mathematical reasoning that it is.[9] The premises that follow are in part a summary of certain points in the theory applicable to a more extended form of the indirect proof.[10]

A. A rational number is one written p/q where q is not equal to 0 and *p* and *q* are positive or negative integers with no common factor other than plus or minus one.

B. An irrational number is one that cannot be written p/q .

C, The square of an even number is also even, and the square of an odd number is odd.

D. A number is even if it has 2 as a factor.

E. The square of an even number is divisible by 4.

The Proof *Per Impossibile* (taking A, B, C, D, and E as premises):

(1) $\sqrt{2} = p/q$	hypothesis	
(2) $2 = p^2/q^2$	(1)	
(3) $p^2 = 2q^2$	(2)	
(4) p^2 is an even number.	(D)	(3)
(5) p is an even number.	(C)	(4)
(6) p^2 is divisible by 4.	(E)	(5)
(7) $2q^2$ is divisible by 4.	(3)	(6)
(8) q^2 is an even number.	(D)	(7)
(9) q is an even number.	(C)	(8)

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[8] *Prior Anal.*, I ,23, 41a23-32 (transl. A. Jenkinson).

[9] R. Courant and H. Robbins, *What Is Mathematics?* (London, 1951), pp. 59-60.

[10] J. Newman, *The World of Mathematics* (New York, 1956), I, 525-27, summing up the work of Richard Dedekind.

(10)	p and q have 2 as a common factor.	(5) (9)
(11)	$-(\sqrt{2} = p/q)$	(A) *Per Impossibile* (Q.E.D.)
(12)	$\sqrt{2}$ is an irrational number.	(B) (11)

Hence the hypothesis that the $\sqrt{2}$ is a *rational* number, since it implies (A) that p/q is in lowest terms with no common factor between *p* and *q* but concludes (10) that both have 2 as a common factor, is necessarily false (11) and its contradictory (12) follows by opposition.[11]

The "syllogistic inference" mentioned by Aristotle is abbreviated in the above sequence by the notations after each step indicating the two premises from which it is derived. For example, in step (4) the complete syllogism would be: A number having 2 as a factor is an even number (D). But p^2 is a number having 2 as a factor (3). Therefore, p^2 is an even number.

* * *

The indirect reduction of syllogisms of the second and third figures to those of the first is an interesting variant of the method of indirect proof or the *reductio ad absurdum.* Aristotle called it the *reductio ad impossibile*[12] and proposed it in summary form as a means of validating all syllogistic moods,[13] even those that admit direct reduction.

Using Aristotle's demonstration as a starting point, Peter the Spaniard (later Pope John XXI—died 1277) worked out a detailed scheme for applying this type of validation in his *Summulae Logicales.* This was the first medieval work proposing to cover the whole of Aristotle's logic systematically. It remained the standard introduction to the subject for more than three hundred years. Forty-eight editions of it came out in the one century that followed the invention of printing. In it are found, for the first time it seems, nearly all the memonic devices later used in teaching logic. Perhaps the best known

[11] Another way of expressing the contradiction is: p/q is in lowest terms (A); it is not the case that p/q is in lowest terms (10).

[12] *Prior Anal.*, I, 5, 27a15; I, 7, 29a35-39.

[13] See *Prior Anal.,* II, 14, 63a.

are the lines *Barbara, Celarent,* etc. which contain in abbreviated form a detailed method of directly reducing the other figures to the first by conversion and interchange of premises.

In the fourth chapter of his *Summulae* Peter—and after him the classical logicians—applied the technique of reducing *per impossibile* to validate Baroco and Bocardo, the two moods of the second and third figures that do not admit direct reduction. The pertinent texts are as follows:

(4.04) Syllogisms must have a mood and a figure. The figure is determined by the arrangement of the three terms as subject and predicate.... The second figure is that in which the same term is predicate in both premises.... The third figure has the same subject in both premises.... The mood is the arranging of the two premises according to quality and quantity in a way suitable [for drawing a conclusion].

(4.11) The second figure has four moods.... The fourth consists of one universal affirmative [premise] and one particular negative and a particular negative conclusion; for example, Every man is an animal. Some stone is not an animal. Therefore, some stone is not a man. This is reducible to the first [figure] *per impossibile.*

(4.12) To reduce *per impossibile* is to infer the contradictory of the second premise from the contradictory of the conclusion together with the first premise. For example, let the above conclusion be contradicted, 'Every stone is a man,' and [then] taken together with the major of the fourth mood as follows: Every man is an animal. Every stone is a man. Therefore, every stone is an animal This conclusion contradicts the minor of the fourth [mood] and a reduction *per impossibile* is effected.

(4.14) The third figure has six moods....

(4.15) The fifth is made up of one particular negative premise and one universal affirmative with a particular negative conclusion, as, Some man is not a stone. Every man is an animal. Therefore, some animal is not a stone. It is reducible to the first [figure] *per impossibile* by taking the contradictory of the conclusion together with the second premise, thus obtaining the contradictory of the remaining [i.e., the first] premise; for example: Every animal is a stone. Every man is an animal. Therefore, every man is a stone.[14] This conclusion, drawn

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[14] A correction for the conclusion, "Therefore, every stone is a man," in the *Summulae.*

from the first mood of the first figure, contradicts the major [premise] of the fifth mood of the third figure.[15]

The "impossibility" for which this kind of reduction is named would consist, as in the forms of indirect proof already examined, in admitting the simultaneous truth of two contradictories. For example, to deny the validity of the fourth mood of the second figure (Baroco) is the same as admitting the truth of the premises but denying the conclusion because of the assumption that the reasoning of this mood is illegitimate. But if the conclusion is false, then its contradictory must be true. This contradictory is then taken with the major premise of the mood in question, which major has already been admitted as true, and a syllogism in the first figure—whose validity is conceded—is formed. The conclusion of this must be admitted as true, because of the admission that the premises are true and the process is valid. But this admittedly true conclusion contradicts the minor, already admitted as true, of the syllogism originally in question. If, then, one denies the validity of Baroco, one is led to affirm contradictories. Therefore the validity of this mood must be admitted.

For Baroco the conclusion which is to be proved as validly drawn from the premises is (Ex) (Sx · — Mx). The predicate variables M, A, and S in the scope of the following premises are abbreviations of the terms used in Peter's illustrations. The proof explained above is then written:

(1)	$(x)(Mx \supset Ax)$	premise
(2)	$(\exists x)(Sx \cdot -Ax)$	premise
(3)	$-(\exists x)(Sx \cdot -Mx)$	hypothesis
(4)	$(x)(Sx \supset Mx)$	(3)
(5)	$(x)(Sx \supset Ax)$	(1) (4)
(6)	$-(\exists x)(Sx \cdot -Ax)$	(5)
(7)	$(\exists x)(Sx \cdot -Ax) \cdot -(\exists x)(Sx \cdot -Ax)$	(2) (6)
(8)	$(\exists x)(Sx \cdot -Mx)$	(1) (2) Per Impossibile

[15] Petri Hispani *Summulae Logicales*, ed. I. Bochenski (Torino, 1947), pp. 37-41.

Similarly, if Bocardo is a valid mood, it will have the conclusion, (Ex) (Ax · — Sx). Contradicting this in the third step of the proof as written above, quantifying it universally as (x) (Ax ⊃ Sx), and taking this universal statement with the minor of Bocardo will give the indirect proof for this mood.

If the propositions in Peter's examples are rearranged according to the principles of exportation, transposition, and partial duality, the process he describes[16] in 4.12 of the above citation is shown in step (5) of the following argument and that of 4.15 in step (9).

From the Third Figure (Bocardo) to the Second (Baroco):

$$(1)\ [(\exists x)(Mx \cdot -Sx)\cdot(x)(Mx \supset Ax)] \supset (\exists x)(Ax \cdot -Sx) \qquad \text{(III: Bocardo)}$$

$$(2)\ [(x)(Mx \supset Ax)\cdot(\exists x)(Mx \cdot -Sx)] \supset (\exists x)(Ax \cdot -Sx) \qquad (1)$$

$$(3)\ (x)(Mx \supset Ax) \supset [(\exists x)(Mx \cdot -Sx) \supset (\exists x)(Ax \cdot -Sx)] \qquad (2)$$

$$(4)\ -[-(\exists x)(Ax \cdot -Sx) \supset -(\exists x)(Mx \cdot -Sx)] \supset -(x)(Mx \supset Ax) \qquad (3)$$

$$(5)\ [(x)(Ax \supset Sx)\cdot(\exists x)(Mx \cdot -Sx)] \supset (\exists x)(Mx \cdot -Ax) \qquad (4)\text{(II: Baroco)}$$

From the Second Figure (Baroco) to the First (Barbara):

$$(5)\ [(x)(Ax \supset Sx)\cdot(\exists x)(Mx \cdot -Sx)] \supset (\exists x)(Mx \cdot -Ax) \qquad \text{(II: Baroco)}$$

$$(6)\ [(\exists x)(Mx \cdot -Sx)\cdot(x)(Ax \supset Sx)] \supset (\exists x)(Mx \cdot -Ax) \qquad (5)$$

$$(7)\ (\exists x)(Mx \cdot -Sx) \supset [(x)(Ax \supset Sx) \supset (\exists x)(Mx \cdot -Ax)] \qquad (6)$$

$$(8)\ -[(x)(Ax \supset Sx) \supset (\exists x)(Mx \cdot -Ax)] \supset -(\exists x)(Mx \cdot -Sx) \qquad (7)$$

$$(9)\ [(x)(Ax \supset Sx)\cdot(x)(Mx \supset Ax)] \supset (x)(Mx \supset Sx) \qquad (8)\text{(I: Barbara)}$$

Thus all three moods imply one another and denying the validity of any one of them requires a like denial of that of the other two.

Indirect demonstration concludes only to the fact that a thing is so or not so and cannot be otherwise, but without giving the intrinsic reason why. Where one must be content with negative grounds for judging, the method is a valuable means of arriving at knowledge. But in some of its more celebrated applications in the history of thought the contradiction

[16] The process is here represented but not the distribution of terms.

to which the unwanted assumption leads is only apparent. Aristotle destroyed the dilemma of Parmenides by the postulate of potentiality, and Zeno's riddles were solved by acknowledging magnitude to be continuous rather than discrete. At the time they were proposed, however, Zeno's arguments raised insoluble difficulties and so opened the way for the scepticism of the Sophists, who were interested in confuting their opponents rather than in getting to the truth.

Socrates used the indirect method of proof to overthrow their easy assumptions and to show the existence of problems that had scarcely occurred to his predecessors. Thinkers of all times have found in the *reductio ad absurdum* and like forms of argumentation a method of clarifying their reflections on basic principles which because of their obscurity or their immediacy cannot be proved directly. In every field of investigation the indirect method will remain useful for its versatility, its apodictic character, and its effectiveness in excluding false hypotheses.